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Iterative Schemes for Multivalued Quasi Variational Inclusions

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Abstract. In this paper, we suggest and analyze a class of iterative schemes for solving multivalued quasi variational inclusions using the resolvent operator method. As special cases, we obtain a number of known and new iterative schems for solving variational inequalities and related optimization problems. The results obtained in this represent an improvement and a significant refinement of previously known results.

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1. Introduction

Multivalued quasi variational inclusion, which was introduced and studied by Noor [l9, 20], is a useful and important extension of the variational principles with a wide range of applications in industry, physical, regional, social, pure and applied sciences. Some special cases have been studied by many authors including Ding [4], Huang [9], Noor [13–15] and Noor and Noor [12]. It is worth mentioning that multival- ued quasi variational inclusions include mixed (quasi) variational inequalities, complementarity problems and nonlinear programming problems as special cases. Quasi variational inclusions provide us with a unified, natural, novel, innovative and general technique to study a wide class of problems arising in different branches of mathematical and engineering sciences, There are a substantial number of numerical methods including projection method and its variant forms, Wiener-Hopf equations, auxiliary principle and descent for solving various classes of variational inequalities and complementarity problems, Noor [16, 17]. It is well known that the projection methods, Wiener-Hopf equations techniques and auxiliary principle techniques cannot be extended and modified for solving variational inclusions. This fact motivated to develop another technique, which involves the use of the resolvent operator associated with maximal monotone operator. In this technique, the given operator is decomposed into the sum of two (maximal) monotone operators. Such a method is known as the operator splitting method. The operator splitting method and related technique have studied by many authors, see, for example, Noor [21] and references therein. Using this technique, one shows that the variational inequalities (inclusions) are equivalent to the fixed point problem. This alternative formulation was used to develop numerical methods for solving various classes of mixed variational inequalities (inclusions) and related problems, see [4, 8–26]. In this paper, we use this alternative formulation to suggest and analyse two-step iterative schemes for multivalued quasi variational inclusions. Our results extend and generalise the previously known results.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively. Let $C(H)$ be a family of all nonempty compact subsets of H. Let $T, V : H \to C(H)$ be the multivalued operators and $g : H \to H$ be a single-valued operator. Let $A(\cdot,\cdot) : H \times H \to H$, consider the problem of finding $u \in H$, $w \in T(u)$, $y \in V(u)$ such that

$$0 \in N(w, y) + A(g(u), u), \tag{2.1}$$

which is called the multivalued quasi variational inclusions, see Noor [19, 20]. Some special cases of (2.1) have been studied by Noor [13–15], Huang [9], Ding [4], Noor–Noor and Rassias [22] and Uko [27] recently. A number of problems arising in structural analysis, mechanics and economics can be studied in the framework of the multivalued quasi variational inclusions; see, for example, [3, 25, 26].

2.1. SPECIAL CASES

I. If $A(\cdot, u) = \partial\phi(\cdot, u) : H \times H \to H$, the subdifferential of a convex, proper and lower semi-continuous function $\phi(\cdot, u)$ with respect to the first argument, then problem (2.1) is equivalent to finding $u \in H$, $w \in T(u)$, $y \in V(u)$ such that

$$\langle N(w, y), v - g(u)\rangle + \phi(v, u) - \phi(g(u), u) \geqslant 0, \quad \text{for all } v \in H, \tag{2.2}$$

which is called the set-valued mixed quasi variational inequality. Problem (2.2) has been studied by Noor [13,14] using the resolvent equations technique.

II. If $A(g(u, v) \equiv A(g(u))$, for all $v \in H$, then problem (2.1) is equivalent to finding $u \in H$, $w \in T(u)$, $y \in V(u)$ such that

$$0 \in N(w, y) + A(g(u)), \tag{2.3}$$

a problem considered and studied by Noor [15] using the resolvent equations technique. See also [11, 27] for the related work.

III. If $A(g(u)) \equiv \partial\phi(g(u))$ is the subdifferential of a proper, convex and lower, semicontinuous function $\phi : H \to R \cup \{+\infty\}$. Then problem (2.1) reduces to: find

$u \in H$, $w \in T(u)$, $y \in V(u)$ such that

$$\langle N(w, y), v - g(u)\rangle + \phi(v) - \phi(g(u)) \geqslant 0. \quad (2.4)$$

Problem (2.4) is known as the set-valued mixed variational inequality and has been studied by Noor–Noor and Rassias [22] and its special cases by Huang [9] and Uko [27].

IV. If the function $\phi(u, v)$ for all $v \in H$, is the indicator function of a closed convex set $K(u)$ in H, that is,

$$\phi(u, v) = K_{(u)}(u) = \begin{cases} 0, & \text{if } u \in K(u) \\ +\infty, & \text{otherwise}, \end{cases}$$

then problem (2.2) is equivalent to finding $u \in H$, $w \in T(u)$, $y \in V(u)$, $g(u) \in K(u)$ such that

$$\langle N(w, y), v - g(u)\rangle \geqslant 0, \quad \text{for all } v \in K(u), \quad (2.5)$$

a problem considered and studied by Noor [18], using the projection method and the implicit Wiener-Hopf equations technique.

V. If $K^*(u) = \{u \in H, \langle u, v\rangle \geqslant 0$, for all $v \in K(u)\}$ is a polar cone of the convex cone $K(u)$ in H, then problem (2.6) is equivalent to finding $u \in H$, $w \in T(u)$, $y \in V(u)$ such that

$$g(u) \in K(u), \quad N(w, y) \in K^*(u) \quad \text{and} \quad \langle N(w, y), g(u)\rangle = 0,$$

which is called the generalized multivalued implicit complementarity problem and appears to be a new one.

For special choices of the operators T, $N(\cdot, \cdot)$, g and the convex set K, one can obtain a large number of complementarity and implicit (quasi) complementarity problems, see, for example, [4, 8–23] and the references therein. We would like to mention that the problem of finding a zero of the sum of two maximal monotone operators, location problem, $\min_{u \in H}\{f(u) + g(u)\}$, where f, g are both convex functions, various classes of variational inequalities and complementarity problems are very special cases of problem (2.1). Thus it is clear that problem (2.1) is general and unifying one and has numerous applications in pure and applied sciences.

We now recall some basic concepts and results.

DEFINITION 2.1 [2]. If T is a maximal monotone operator on H, then, for a constant $\rho > 0$, the resolvent operator associated with T is defined by

$$J_T(u) = (I + \rho T)^{-1}(u), \quad \text{for all } u \in H,$$

where I is the identity operator. It is known that the monotone operator T is maximal monotone if and only if the resolvent operator J_T is defined everywhere on the space. Furthermore, the resolvent operator J_T is single-valued and nonexpansive.

REMARK 2.1. Since the operator $A(\cdot,\cdot)$ is a maximal monotone operator with respect to the first argument, for a constant $\rho > 0$, we denote by

$$J_{A(u)} \equiv (I + \rho A(u))^{-1}(u), \quad \text{for all } u \in H,$$

the resolvent operator associated with $A(v,u) \equiv A(u)$ for all $v \in H$. For example, if $A(v,u) = \partial\phi(v,u)$, for all $u, v \in H$, and $\phi(v,u) : H \times H \to H$ is a proper, convex and lower semicontinuous with respect to the first argument, then it is well-known that $\partial\phi(v,u)$ is a maximal monotone operator with respect to the first argument. In this case, the resolvent operator $J_{A(u)} = J_{\partial\phi(u)}$ is

$$J_{\partial\phi(u)} = (I + \rho\partial\phi(\cdot,u))^{-1}(u) = (I + \rho\partial\phi(u))^{-1}(u) \quad \text{for all} u \in H,$$

which is defined everywhere on the space H, where $\partial\phi(u) \equiv \partial\phi(v,u)$ for all $v \in H$. For a recent state-of-the-art of the convex analysis, see Gao [5].

DEFINITION 2.2. For all $u_1, u_2 \in H$, the operator $N(\cdot,\cdot)$ is said to be strongly monotone and Lipschitz continuous with respect to the first argument, if there exist constants $\alpha > 0$, $\beta > 0$ such that

$$\langle N(\cdot,w_1) - N(\cdot,w_1), u_1 - u_2\rangle \geqslant \alpha\|u_1 - u_2\|^2, \quad \text{for all } w_1 \in T(u_1),\, w_2 \in T(u_2)$$
$$\mu\|N(u_1,\cdot) - N(u_2,\cdot)\| \leqslant \beta\|u_1 - u_2\|.$$

In a similar way, we can define strong monotonicity and Lipschitz continuity of the operator $N(\cdot,\cdot)$ with respect to second argument.

DEFINITION 2.3. The set-valued operator $V : H \to C(H)$ is said to be M-Lipschitz continuous, if there exists a constant $\xi > 0$ such that

$$M(V(u), V(v)) \leqslant \xi\|u - v\|, \quad \text{for all } u, v \in H,$$

where $M(\cdot,\cdot)$ is the Hausdorff metric on $C(H)$.

We also need the following condition.

ASSUMPTION 2.1. For all $u, v, w \in H$, the resolvent operator $J_{A(u)}$ satisfies the condition

$$\|J_{A(u)}w - J_{A(v)}w\| \leqslant \nu\|u - v\|,$$

where $\nu > 0$ is a constant.

Assumption 2.1 is satisfied when the operator A is monotone jointly with respect to two arguments. In particular, this implies that A is monotone with respect to first argument, see Noor [20].

3. Main Results

In this section, we use the resolvent operator technique to establish the equivalence between the multi-valued quasi variational inclusions and the implicit resolvent fixed points. This equivalence is used to suggest an iterative method for solving the quasi variational inclusions. For this purpose, we need the following result.

LEMMA 3.1. (u, w, y) *is a solution of (2.1) if and only if* (u, w, y) *satisfies the relation*

$$g(u) = J_{A(u)}[g(u) - \rho N(w, y)], \tag{3.1}$$

where $\rho > 0$ *is a constant and* $J_{A(u)} = (I + \rho A(u))^{-1}$ *is the resolvent operator.*

Proof. Let $u \in H$, $w \in T(u)$, $y \in V(u)$ be a solution of (2.1). Then, for a constant $\rho > 0$,

$$\begin{aligned}(2.1) &\Longleftrightarrow 0 \in \rho N(w, y) + \rho A(g(u), u)\\ &\Longleftrightarrow 0 \in -(g(u) - \rho N(w, y)) + (I + \rho A(u))g(u)\\ &\Longleftrightarrow g(u) = J_{A(u)}[g(u) - \rho N(w, y)],\end{aligned}$$

the required result. □

From Lemma 3.1, we conclude that the multivalued quasi variational inclusions (2.1) are equivalent to the implicit fixed point problem (3.1). This alternative formulation is very useful from both theoretical and numerical analysis points of view. We use this equivalence to propose some iterative algorithm for solving quasi variational inclusions (2.1) and related problems.

The relation (3.1) can be written as

$$u = u - g(u) + J_{A(u)}[g(u) - \rho N(w, u))], \tag{3.2}$$

where $\rho > 0$ is a constant.

This fixed point formulation allows us to suggest the following unified iterative algorithm.

ALGORITHM 3.1. Assume that $T, V : H \to C(H)$, $g : H \to H$ and $N(\cdot, \cdot)$, $A(\cdot, \cdot) : H \times H \to H$ are operators. For a given $u_0 \in H$, compute the sequences $\{u_n\}$, $\{w_n\}$, $\{y_n\}$, $\{\overline{w_n}\}$ and $\{\overline{y_n}\}$ by the iterative schemes

$$w_n \in T(u_n) : \|w_{n+1} - w_n\| \leqslant M(T(u_{n+1}), T(u_n)) \tag{3.3}$$

$$y_n \in V(u_n) : \|y_{n+1} - y_n\| \leqslant M(V(u_{n+1}), V(u_n)) \tag{3.4}$$

$$\overline{w_n} \in T(z_n) : \|\overline{w_{n+1}} - \overline{w_n}\| \leqslant M(T(z_{n+1}), T(z_n)) \tag{3.5}$$

$$\overline{y_n} \in V(z_n)) : \|\overline{y_{n+1}} - \overline{y_n}\| \leqslant M(V(z_{n+1}), V(z_n)) \tag{3.6}$$

$$z_n = (1 - \beta_n)u_n + \beta_n\{u_n - g(u_n) + J_{A(u_n)}[g(u_n) - \rho N(w_n, y_n)]\} \tag{3.7}$$

$$\begin{aligned}u_{n+1} = (1 - \alpha_n)u_n + \alpha_n\{z_n - g(z_n) + J_{A(z_n)}[g(z_n) - \rho N(\overline{w_n}, \overline{y_n})]\},\\ n = 0, 1, 2, \ldots ,\end{aligned} \tag{3.8}$$

where $0 \leqslant \alpha_n, \beta_n \leqslant 1$; for all $n \geqslant 0$, and $\sum_{n=0}^{\infty} \alpha_n$ diverges and $\rho > 0$ is constant. Algorithm 3.1 is similar to the Ishikawa iterative scheme for solving quasi variational inclusions. For $\beta_n = 0$ and $\alpha_n = \lambda$, Algorithm 3.1 has been studied by Noor [19,20].

If $A(\cdot, v) \equiv \partial\phi(\cdot, v)$ for all $v \in H$, is an indicator function of a closed convex set $K(u)$ in h, then $J_{A(u)} \equiv P_{K(u)}$, the projection of H onto the convex set $K(u)$ in H. Consequently, Algorithm 3.1 collapses to:

ALGORITHM 3.2. For given $u_0 \in H$, $w_0 \in T(u_0)$, $y_0 \in V(u_0)$, $g(u_0) \in K(u_0)$, compute the sequences $\{u_n\}$, $\{w_n\}$ and $\{y_n\}$ from the iterative schemes

$$
\begin{aligned}
&w_n \in T(u_n) : \|w_{n+1} - w_n\| \leqslant M(T(u_{n+1}), T(u_n)) \\
&\overline{w_n} \in T(z_n) : \|\overline{w_{n+1}} - \overline{w_n}\| \leqslant M(T(z_{n+1}), T(z_n)) \\
&\overline{y_n} \in V(z_n)) : \|\overline{y_{n+1}} - \overline{y_n}\| \leqslant M(V(z_{n+1}), V(z_n)) \\
&y_n \in V(u_n) : \|y_{n+1} - y_n\| \leqslant M(V(u_{n+1}), V(u_n)) \\
&z_n = (1 - \beta_n)u_n + \beta_n\{u_n - g(u_n) + P_{K(u_n)}[g(u_n) - \rho N(w_n, y_n)]\} \\
&u_{n+1} = (1 - \alpha_n)u_n + \alpha_n\{z_n - g(z_n) + P_{K(z_n)}[g(z_n) - \rho N(\overline{w_n}, \overline{y_n})]\}, \\
&\quad n = 0, 1, 2, \dots ,
\end{aligned}
$$

where $0 < \alpha_n, \beta_n < 1$, for all $n \geqslant 0$ and $\sum_{n=0}^{\infty} \alpha_n$ diverges. Algorithm 3.2 appears to be a new one for multivalued variational inequalities (2.5).

For suitable and appropriate choice of the operators T, V, g and the spaces H, K; one can obtain a number of algorithms for solving variational inclusions and related problems.

THEOREM 1. *Let the operator $N(\cdot, \cdot)$ be strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$ with respect to the first argument. Let $g : H \to H$ be strongly monotone with constant $\delta > 0$ and be Lipschitz continuous with constant $\delta > 0$. Assume that the operator $N(\cdot, \cdot)$ is Lipschitz continuous with constant $\eta > 0$ with respect to the second argument and V is M-Lipschitz continuous with constant $\xi > 0$. Let $T : H \to C(H)$ be a M-Lipschitz continuous with constant $\mu > 0$. If Assumption 2.1 holds and*

$$\left|\rho - \frac{\alpha - (1-k)\eta\xi}{\beta^2\mu^2 - \eta^2\xi^2}\right| < \frac{\sqrt{[\alpha - (1-k)\eta\xi]^2 - k(\beta^2\mu^2 - \eta^2\xi^2)(2-k)}}{\beta^2\mu^2 - \eta^2\xi^2} \tag{3.9}$$

$$\alpha > (1-k)\eta\xi + \sqrt{k(\beta^2\mu^2 - \eta^2\xi^2)(2-k)} \tag{3.10}$$

$$\rho\eta\xi < 1 - k \tag{3.11}$$

$$k = 2(\sqrt{1 - 2\sigma + \delta^2}) + \nu, \tag{3.12}$$

then there exist $u \in H$, $w \in T(u)$, $y \in V(u)$ satisfying the multivalued quasi variational inclusions (2.1) and the sequences $\{u_n\}$, $\{w_n\}$, $\{y_n\}$, $\{\overline{w_n}\}$ and $\{\overline{y_n}\}$ generated by Algorithm 3.1 converges to u, w, y, $\overline{w}$ and $\overline{y}$ strongly in H respectively.

Proof. If the Assumption 2.1 and the conditions (3.9)–(3.11) hold, then it has been shown in [19, Theorem 3.1, pp. 106] that there exists a solution $u \in H$, $w \in T(u)$, $y \in V(u)$ satisfying the variational inclusion (2.1). Let $u \in H$ be the solution of (2.1). Then

$$\begin{aligned} u &= (1-\alpha_n)u + \alpha_n\{u - g(u) + J_{A(u)}[g(u) - \rho N(w, y)]\} && (3.13)\\ &= (1-\beta_n)u + \beta_n\{u - g(u) + J_{A(u)}[g(u) - \rho N(w, y)]\}. && (3.14)\end{aligned}$$

From (3.8), (3.13) and Assumption 2.1, we have

$$\begin{aligned} \|u_{n+1} - u\| &\leqslant (1-\alpha_n)\|u_n - u_{n-1}\| + \alpha_n\|z_n - u - (g(z_n) - g(u))\| \\ &\quad + \alpha_n\|J_{A(z_n)}[g(z_n) - \rho N(\overline{w_n}, \overline{y_n})] - J_{A(u)}[g(u) - \rho N(w, y)]\| \\ &\leqslant (1-\alpha_n)\|u_n - u_{n-1}\| + \alpha_n\|z_n - u - (g(z_n) - g(u))\| \\ &\quad + \alpha_n\|J_{A(z_n)}[g(z_n) - \rho N(\overline{w_n}, \overline{y_n})] - J_{A(u)}[g(z_n) - \rho N(\overline{w_n}, \overline{y_n})]\| \\ &\quad + \alpha_n\|J_{A(u)}[g(z_n) - \rho N(\overline{w_n}, \overline{y_n})] - J_{A(u)}[g(u) - \rho N(w_n, y)]\| \\ &\leqslant (1-\alpha_n)\|u_n - u\| + 2\alpha_n\|z_n - u - (g(z_n) - g(u))\| \\ &\quad + \alpha_n\|z_n - u - \rho\{N(\overline{w_n}, \overline{y_n}) - N(w, \overline{y_n})\}\| \\ &\quad + \alpha_n\rho\|N(w, \overline{y_n}) - N(w, y)\| + \alpha_n\nu\|z_n - u\|. \qquad (3.15)\end{aligned}$$

Since $g : H \to H$ is strongly monotone Lipschitz continuous, we have

$$\begin{aligned} &\|z_n - u - (g(z_n) - g(u))\|^2 \\ &\quad = \|z_n - u\|^2 - 2\langle z_n - u, g(z_n) - g(u)\rangle + \|g(z_n) - g(u)\|^2 \\ &\quad \leqslant (1 - 2\sigma + \delta^2)\|z_n - u\|^2. \qquad (3.16)\end{aligned}$$

Since $N(\cdot, \cdot)$ is a strongly monotone Lipschitz continuous operator with respect to first argument, it follows that

$$\begin{aligned} &\|z_n - u - \rho\{N(\overline{w_n}, \overline{y_n}) - N(w, \overline{y_n})\}\|^2 \\ &\quad = \|z_n - u\|^2 - 2\rho\langle N(\overline{w_n}, \overline{y_n}) - N(w, \overline{y_n}), z_n - u\rangle \\ &\qquad + \rho^2\|N(\overline{w_n}, \overline{y_n}) - N(w, \overline{y}_n)\|^2 \\ &\quad \leqslant (1 - 2\rho\alpha + \rho^2\beta^2\mu^2)\|z_n - u\|^2. \qquad (3.17)\end{aligned}$$

Using the Lipschitz continuity of the operator $N(\cdot, \cdot)$ with respect to second argument and the M-Lipschitz continuity of V, for all $w \in T(u)$, $y \in V(u)$, we have

$$\begin{aligned} \|N(w, \overline{y_n}) - N(w, y)\| &\leqslant \eta\|\overline{y_n} - y\| \\ &\leqslant \eta M(V(z_n), V(u)) \\ &\leqslant \eta\xi\|z_n - u\|. \qquad (3.18)\end{aligned}$$

From (3.15), (3.16), (3.17) and (3.18), we have

$$\begin{aligned}\|u_{n+1}-u\| &\leqslant \alpha_n\{2(\sqrt{1-2\sigma+\delta^2})+\nu+\rho\eta\xi \\ &\quad+\sqrt{1-2\rho\alpha+\rho^2\beta^2\mu^2}\}\|z_n-u\| \\ &\quad+(1-\alpha_n)\|u_n-u\| \\ &=(1-\alpha_n)\|u_n-u\|+\alpha_n\|(k+\rho\eta\xi+t(\rho))\}\|z_n-u\|, \\ &=(1-\alpha_n)\|u_n-u\|+\alpha_n\theta\|z_n-u\|, \end{aligned} \tag{3.19}$$

where

$$\theta = k+\rho\eta\xi+t(\rho) \tag{3.20}$$

$$t(\rho)=\sqrt{1-2\rho\alpha+\rho^2\beta^2\mu^2} \tag{3.21}$$

In a similar way, from (3.7) and (3.14), we obtain

$$\begin{aligned}\|z_n-u\| &\leqslant (1-\beta_n)\|u_n-u\|+\beta_n\{2\|u_n-u-(g(u_n)-g(u)\| \\ &\quad+\|u_n-u-\rho(N(w_n,y_n)-\rho N(w,y))\|\} \\ &\leqslant (1-\beta_n)\|u_n-u\|+\beta_n(k+\rho\xi\eta+t(\rho))\|u_n-u\|, \\ &\qquad \text{using (3.12), (3.18) and (3.21).} \\ &=(1-\beta_n)\|u_n-u\|+\beta_n\theta\|u_n-u\|, \quad \text{using (3.20)}, \\ &\leqslant \|u_n-u\|. \end{aligned} \tag{3.22}$$

Combining (3.19) and (3.22), we obtain

$$\begin{aligned}\|u_{n+1}-u\| &\leqslant \{(1-\alpha_n(1-\theta))\}\|u_n-u\| \\ &=\prod_{i=0}^{n}[1-(1-\theta)\alpha_i]\|u_0-u\|. \end{aligned} \tag{3.23}$$

From (3.9)–(3.11), it follows that $\theta < 1$. Since $\sum_{n=0}^{\infty}\alpha_n$ diverges and $1-\theta \geqslant 0$, it follows that $\prod_{i=0}^{\infty}[1-(1-\theta)\alpha_i]=0$. Hence the sequence $\{u_n\}$ converges strongly to u. From (3.22), we see that the sequence $\{z_n\}$ also converges strongly to u. From (3.18), it follows that the sequence $\{y_n\}$ is a Cauchy sequence in H, so there exists a $y \in H$ such that $y_n \to y$. In a similar way, for all $w \in T(u)$.

$$\|\overline{w_n}-w\| \leqslant M(T(z_n),T(u)) \leqslant \mu\|z_n-u\|,$$

which implies that the sequence $\{\bar{w}_n\}$ is a Cauchy sequence, that is, there exists $w \in H$ such that $\bar{w}_n \to w$. Now by using the continuity of the operators, T, V, g, $J_{A(u)}$ and Algorithm 3.1, we have

$$g(u)=J_{A(u)}[g(u)-\rho N(w,y)] \in H.$$

It remains to show that $y \in V(u)$, $w \in T(u)$, $\bar{w} \in T(z)$ and $\bar{y} \in V(z)$. In fact,

$$\begin{aligned} d(w, T(u)) &\leqslant \|w - w_n\| + d(w_n, T(u)) \\ &\leqslant \|w - w_n\| + M(T(u_n), T(u)) \\ &\leqslant \|w - w_n\| + \mu\|u_n - u\| \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

where $d(w, T(u)) = \inf\{\|w - z\| : z \in T(u)\}$. Since the sequences $\{w_n\}$ and $\{u_n\}$ are the Cauchy sequences, it follows that $d(w, T(u)) = 0$. By invoking Lemma 3.1, we have $u \in H$, $w \in T(u)$, $y \in V(u)$, which satisfies the quasi variational inclusion (2.1) and consequently $u_n \to u$, $\overline{w_n} \to \bar{w}$, $\overline{y_n} \to \bar{y}$, $w_n \to w$ and $y_n \to y$ in H strongly, the required result. □

REMARK 3.1. It is worth mentioning that the assumption 2.1 and the conditions 3.9–3.11, which play an important part in the derivation of the main result, that is, Theorem 3.1, are very convenient and reasonable easy to verify in practical problems, see Noor [14, 20]. For different choice of the operators T, V, $N(\cdot, \cdot)$, and $A(\cdot, \cdot)$, these conditions are well known and have been already used in the existence of a solution of variational inequalities and inclusions.

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